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ON EXTREMAL PROPERTIES OF CIRCULAR SLIT COVERING SURFACES

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1. Let B be a non-compact domain of finite connectivity on the z -plane. We suppose that each component C_j ($j=1, \dots, N$) of its boundary C is a continuum. Let z_j and ζ_k ($j=1, \dots, \iota$; $k=1, \dots, \kappa$; $\iota \geq 1, \kappa \geq 1$) be arbitrarily preassigned $\iota + \kappa$ points in B , and m_j and n_k ($j=1, \dots, \iota$; $k=1, \dots, \kappa$) be arbitrarily preassigned positive integers under the condition

$$(1) \quad p \equiv \sum_{j=1}^{\iota} m_j = \sum_{k=1}^{\kappa} n_k.$$

We shall conventionally agree to take as $\zeta_1 = \infty \in B$ through the present paper. Let \mathfrak{F}_p be the class of analytic functions $w=f(z)$ on B with the following properties:

- (a) f has the only zeros z_j ($j=1, \dots, \iota$) and the only poles ζ_k ($k=1, \dots, \kappa$) with their orders m_j and n_k , respectively;
- (b) The rotation number of the image of each C_j ($j=1, \dots, N$) about $w=0$ under f is equal to zero; i. e.

$$\nu_j(f) \equiv \frac{1}{2\pi} \int_{C_j^*} d \arg f = 0 \quad (j=1, \dots, N),$$

where C_j^* ($j=1, \dots, N$) are simple analytic closed curves homotopic to C_j in $B - \sum_{j=1}^{\iota} \{z_j\} - \sum_{k=1}^{\kappa} \{\zeta_k\}$ and $\nu_j(f)$ ($j=1, \dots, N$) are integers not depending on a particular choice of C_j^* ;

$$(c) \quad \left| \int_C \lg |f| d \arg f \right| < +\infty,$$

where the line integral means $\lim_{n \rightarrow \infty} \int_{\partial B_n} \lg |f| d \arg f$ with an exhaustion $\{B_n\}_{n=1}^{\infty}$ of B ;

- (d) f satisfies the normalization condition

$$\lim_{z \rightarrow \infty} \frac{f(z)}{z^p} = 1.$$

2. Let

$$(2) \quad J(f) = \int_C \lg |f| d \arg f - 2\pi \sum_{j=1}^{\iota} m_j \lg |f^{[m_j]}(z_j)| - 2\pi \sum_{k=1}^{\kappa} n_k \lg |f^{[n_k]}(\zeta_k)|$$

for $f \in \mathfrak{F}_p$, where

$$f^{(m_j)}(z_j) = \lim_{z \rightarrow z_j} \frac{f(z)}{(z - z_j)^{m_j}} = \frac{1}{m_j!} f^{(m_j)}(z_j) \quad (j=1, \dots, \iota),$$

$$f^{(n_k)}(\zeta_k) = \lim_{z \rightarrow \zeta_k} \frac{1}{(z - \zeta_k)^{n_k}} f(z) = \frac{1}{n_k!} \left[\left(\frac{1}{f(z)} \right)^{(n_k)} \right]_{z=\zeta_k} \quad (k=2, \dots, \kappa).$$

Then, the theorem 1 immediately follows from the theorem of [2].

THEOREM 1. *There exists the unique element φ in \mathfrak{F}_p which minimizes $J(f)$ on \mathfrak{F}_p . Further φ is the unique element of \mathfrak{F}_p which maps B onto the p -sheeted covering surface of which the boundary consists of circular slits centred at the origin on the basic w -plane.*

3. Let \mathfrak{F}_p' be the subclass of \mathfrak{F}_p which consists of functions $f(z)$ of \mathfrak{F}_p satisfying the condition

$$(3) \quad \int_G \lg |f| d \arg f \leq 0.$$

Then \mathfrak{F}_p' is not vacuous, for there exists a rational function on the z -plane with the properties. Let

$$(4) \quad I(f) = \prod_{j=1}^{\iota} |f^{(m_j)}(z_j)|^{m_j} \prod_{k=2}^{\kappa} |f^{(n_k)}(\zeta_k)|^{n_k}$$

for $f \in \mathfrak{F}_p'$. Then, we have the following theorem.

THEOREM 2. *The function φ of the theorem 1 is the unique element in \mathfrak{F}_p' which maximizes $I(f)$ on \mathfrak{F}_p' .*

Proof. It is immediately seen that there holds

$$(5) \quad \int_G \lg |\varphi| d \arg \varphi = 0$$

for φ of the theorem 1 and thus $\varphi \in \mathfrak{F}_p'$. We note that

$$(6) \quad J(f) = \int_G \lg |f| d \arg f - 2\pi \lg I(f)$$

for any $f \in \mathfrak{F}_p'$. Then, by (5), (6) and theorem 1, there holds

$$-2\pi \lg I(\varphi) = J(\varphi) \leq J(f) \leq -2\pi \lg I(f)$$

and thus

$$(7) \quad I(\varphi) \geq I(f)$$

for φ of the theorem 1 and any $f \in \mathfrak{F}_p'$. Further, by the theorem 1, the equality in (7) occurs if and only if $f(z) \equiv \varphi(z)$.

In the case $p=1$ in 1, we know that $\kappa=1$, $m_1=n_1=1$ and

$$(8) \quad I(f) = |f'(z_1)|.$$

Thus we have the following corollary of the theorem 2.

COROLLARY. *There exists a unique element φ in \mathfrak{F}_1' which maximizes $|f'(z_1)|$ on \mathfrak{F}_1' . Further, φ is the unique element of \mathfrak{F}_1' which univalently maps B onto the domain of which the boundary consists of circular slits centred at the origin.*

4. Let \mathfrak{F}_p'' be the subclass of \mathfrak{F}_p which consists of functions $f(z)$ of \mathfrak{F}_p being p -valent.

LEMMA. $\mathfrak{F}_p'' \subset \mathfrak{F}_p'$.

Proof. Let $f(z)$ be an arbitrary element of \mathfrak{F}_p'' , let F be the image covering surface of B by the mapping $w=f(z)$ and let Γ be the boundary of F . We can take a sufficiently small positive number r such that Γ does not lie over $|w| \leq r$ and $|w| \geq 1/r$. Let F_r be the subset of F obtained by taking off from F the portions of F over $|w| \leq r$ and $|w| \geq 1/r$. Then, by the green's formula, we have

$$(9) \quad \begin{aligned} D_{F_r}(\lg|w|) &= \int_{\partial F_r} \lg|w| d \arg w \\ &= \int_{\Gamma} \lg|w| d \arg w - 4\pi p \lg r \\ &= \int_c \lg|f| d \arg f - 4\pi p \lg r, \end{aligned}$$

where $D_{F_r}(\lg|w|)$ is the Dirichlet's integral of $\lg|w|$ on F_r . On the other hand,

$$(10) \quad D_{F_r}(\lg|w|) \leq p D_{(r < |w| < 1/r)}(\lg|w|) = -4\pi p \lg r,$$

for $f(z)$ is p -valent. By (9) and (10), we have

$$\int_c \lg|f| d \arg f \leq 0$$

and thus $f \in \mathfrak{F}_p'$.

We note that $\varphi \in \mathfrak{F}_p''$ for φ of the theorem 1. Then, by the theorem 2 and the lemma, we have immediately the following theorem.

THEOREM 3. *The function φ of the theorem 1 is the unique element in \mathfrak{F}_p'' which maximizes $I(f)$ on \mathfrak{F}_p'' .*

We note that \mathfrak{F}_1'' consists of all univalent functions $f(z)$ on B which satisfy the conditions

$$f(z_1)=0, \quad f(\infty)=\infty, \quad f'(\infty)=1.$$

Then we have the following classical theorem as the corollary of the theorem 3 (cf. [1], [3]).

COROLLARY. *The function φ of the corollary of the theorem 2 is the unique element in \mathfrak{F}_1'' which maximizes $|f'(z_1)|$ on \mathfrak{F}_1'' .*

5. Example 1. \mathfrak{F}_p'' is a strict subclass of \mathfrak{F}_p' ; i. e. $\mathfrak{F}_p'' \subsetneq \mathfrak{F}_p'$. To see this, it is sufficient to show that there exists even the function of \mathfrak{F}_p' of which the valence is *not bounded*.

Let \mathcal{A} be the covering surface over the ω -plane obtained as the image of the angular domain

$$\left\{ \zeta \left| \frac{3}{4}\pi < \arg \zeta < \frac{5}{4}\pi \right. \right\}$$

by the mapping $\omega = e^\zeta$. Then, the valence of \mathcal{A} over ω -plane is not bounded. Let G be the region obtained from the strip region

$$\left\{ \omega \left| -\frac{\pi}{2}i \leq \Im \omega \leq \frac{3}{2}\pi i \right. \right\}$$

by taking off the closed disk

$$\left\{ \omega \mid |\omega - \pi i| \leq r \right\} \quad \left(\frac{1}{\sqrt{2\pi}} \leq r < \frac{\pi}{2} \right).$$

Let \mathcal{A}' and G' be \mathcal{A} and G slit along the segment

$$l = \left\{ \omega \left| \Re \omega = \frac{1}{\sqrt{2}} e^{-\pi/4}, \quad -\frac{1}{2} e^{-\pi/4} \leq \Im \omega \leq \frac{1}{2} e^{\pi/4} \right. \right\},$$

respectively, F_ω be the covering surface over the ω -plane obtained by the crosswise connection of \mathcal{A}' and G' along the common slit l and F' be the image covering surface of F_ω by the mapping $w = e^\omega$. Then we obtain the covering surface F over the w -plane from F' by the identification along both side of the boundary component over the negative imaginary axis of F' . F is the double connected planar covering surface over the w -plane. Thus we can conformally map F onto the schlicht domain B of

which the boundary consists of the circular slits centred at the origin and further can take B and the mapping function $z=g(w)$ such that the conditions

$$g(0)=0, \quad g(\infty)=\infty, \quad g'(\infty)=1$$

is satisfied. The inverse function $w=f(z)\equiv g^{-1}(z)$ maps B onto F under the condition

$$f(0)=0, \quad f(\infty)=\infty, \quad f'(\infty)=1.$$

It is immediately verified that the function $f(z)$ belongs to \mathfrak{F}_1' . In fact,

$$\begin{aligned} \int_{c_1} \lg |f| d \arg f &= 2 \int_0^\infty d\beta \int_0^{\varepsilon^{-\beta}} r dr = \frac{1}{2}, \\ \int_{c_2} \lg |f| d \arg f &= -\pi r^2 \leq -\frac{1}{2} \end{aligned}$$

and then

$$\int_c \lg |f| d \arg f \leq 0,$$

where C_1 (or C_2) is the boundary component of B on the right (or left) half plane, respectively. However the function $f(z)$ cannot belong to \mathfrak{F}_1'' , because the valence of $f(z)$ is not bounded.

By the analogy the present example we can infer that *any* class \mathfrak{F}_p' contains the functions of which the valences are not bounded and \mathfrak{F}_p' is a large class in comparison with \mathfrak{F}_p'' . The theorem 2 asserts that φ preserves the extremality with respect to the functional $I(f)$ even on such the class \mathfrak{F}_p' . For instance, by the corollary of the theorem 2 there holds

$$|f'(0)| < 1$$

for the function $f(z)$ of the present example, for the extremal function $\varphi(z)$ in the present case is $\varphi(z) \equiv z$.

6. Example 2. Does the function φ of the theorem 1 preserve the extremality with respect to the functional $I(f)$ on the class \mathfrak{F}_p ? The following example gives the negative answer for this question.

Let Δ be the domain

$$\{w | e^{-\varepsilon} < |w| < e^\varepsilon, -(\alpha + \varepsilon) < \arg w < \alpha + \varepsilon\} \quad (0 < \alpha < \pi - \varepsilon, \varepsilon > 0),$$

and G be the whole w -plane. Let Δ' and G' be Δ and G slit along the circular arc

$$l = \{w | |w| = 1, -\alpha \leq \arg w \leq \alpha\},$$

and F be the covering surface over the w -plane obtained by the crosswise connection of A' and G' along the common slit l . Then F is the simply-connected planar surface. Thus we can map F onto the schlicht domain B of which the boundary is a circular slit centred at the origin and further can take B and the mapping function $z=g(w)$ such that the conditions

$$g(0)=0, \quad g(\infty)=\infty, \quad g'(\infty)=1$$

is satisfied. The inverse function $w=f(z)\equiv g^{-1}(z)$ maps B onto F under the condition

$$f(0)=0, \quad f(\infty)=\infty, \quad f'(\infty)=1.$$

It is obvious that $f(z) \in \mathfrak{F}_1$. However $f(z) \notin \mathfrak{F}_1'$, for

$$\int_0^1 \lg |f| d \arg f = \int_{\partial A} \lg |w| d \arg w = 4\epsilon(\alpha + \epsilon) > 0.$$

Let B^* be the image domain of G' by $g(w)$. Then we see that $\bar{B}^* \subset B$ and the restriction of $f(z)$ on B^* is the mapping function of B^* onto the domain G' of which the boundary is the circular slit l . Thus, by the corollary of the theorem 3, we have

$$|f'(0)| > 1.$$

On the other hand, $\varphi(z) \equiv z$ and thus $\varphi'(0) = 1$ for the present B . Consequently, we see that

$$|f'(0)| > \varphi'(0),$$

which rejects the extremality of $\varphi(z)$ with respect to $I(f)$ on the class \mathfrak{F}_1 .

By the analogy of the present example, we can infer that the function φ of the theorem 1 does not preserve the extremality with respect to the functional $I(f)$ on any class \mathfrak{F}_p .

7. In the next paper, we shall concern ourselves with the conformal mappings onto other types of canonical slit covering surfaces and their extremality.

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